Three-Dimensional Motion and Stability of Two Rotating Cable-Connected Bodies

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The three-dimensional linearized equations of motion for a cable-connected space station-counter-weight system are developed using a Lagrangian formulation. The system model employed allows for cable and end body damping and restoring effects. To first degree, the out-of-plane equations uncouple from the in-plane equations. From the general in-plane characteristic equation, necessary conditions for stability are obtained. The Routh-Hurwitz necessary and sufficient conditions for stability are derived for the general out-of-plane characteristic equation. Special cases of the in-plane and out-of-plane equations are then examined for stability. Time constants for the least damped mode are obtained for a range of system parameters by numerical examination of the roots of the in-plane and out-of-plane characteristic polynomials. For the in-plane case, a comparison with results previously obtained in a two-dimensional treatment (but with a different damping scheme) is made. A typical transient response is simulated numerically.

Nomenclature

A = coordinate system moving with the local vertical and located at the cm of the system; unit vectors $\hat{a}_i, i=1,2,3$

$B(C)$ = coordinate system fixed in body 1 (2) at its cm whose axes are principal axes of body 1 (2); unit vectors $\hat{b}_i(\hat{c}_i)$

cm = center of mass

$c_{B_i}(c_{C_i})$ = rotational spring constant for a restoring torque about the $B_i(C_i)$ axis

D = coordinate system located at the system cm but with its first ordered axis along the cable line; unit vectors $\hat{d}_i$

$\mathfrak{F}$ = Rayleigh dissipation function

$I_{B_i}(I_{C_i})$ = moment of inertia of body 1 (2) about the B_i (C_i) axis

$k_{B_i}(k_{C_i})$ = rotational damping constant for a torque due to friction about the $B_i(C_i)$ axis

$k_1(k_2)$ = cable restoring (damping) constant

$\mathcal{L}, \mathcal{L}_0, \mathcal{L}_e$ = instantaneous, unstretched, or equilibrium (respectively) cable length

m_i = mass of the end body, $i=1,2$

s = nominal inertial system spin rate, $\dot{\theta}_n + \Omega$

T = total kinetic energy

t = time

V = total potential energy

$\alpha_1(\alpha_2)$ = coordinate measuring the variation of $\beta_3(\gamma_3)$ from its equilibrium value

$\beta_i(\gamma_i)$ = ith angle in a 1-2-3 rotational sequence used to describe the orientation of body 1(2) with respect to the A system

Γ = ratio: $\Omega/(\dot{\theta}_n+\Omega)$

δ = dimensionless coordinate measuring the variation of $\mathcal{L}$ from its equilibrium value $\delta=(\mathcal{L}-\mathcal{L}_e)/\mathcal{L}_e$

θ_1 = angle in the orbit plane measuring the orientation of the cable line with respect to the A system

θ_2 = angle measuring the out-of-plane orientation of the cable with respect to the A system

$\dot{\theta}_n$ = nominal spin rate of the system; also the equilibrium value of $\dot{\beta}_3$ and $\dot{\gamma}_3$

λ = eigenvalue of a characteristic equation

μ = reduced mass of the system = $m_1 m_2/(m_1+m_2)$

$\rho_1(\rho_2)$ = attachment length of body 1(2) (distance from the cm of body to the point of cable attachment)

T = time constant associated with the least damped mode

τ = nondimensional time, $\tau=st$

χ = deviation of θ_1 from equilibrium value

Ω = orbital angular velocity of the system cm

$\omega_{B_i}(\omega_{C_i})$ = $B_i(C_i)$ component of the angular velocity of body 1 (2)

$\bar{r}_{i/0}$ = geocentric position vector of the cm of body $i(i=1,2)$

$\bar{r}_{i/A}$ = vector from the system cm to the cm of body $i(i=1,2)$

$\bar{\mathcal{L}}$ = vector from the attachment point of body 2 to the attachment point of body 1

$\bar{r}_{A/0}$ = geocentric position vector of the system cm

$\bar{r}_{1/P}(\bar{r}_{2/P})$ = vector from the attachment point of body 1 (2) to the cm of body 1 (2)

$\bar{\omega}_{B/A}$ = angular velocity of the B system with respect to the A system

$\bar{\omega}_{A,B,C,D/F}$ = angular velocity of the $A,B,C,$ or D systems with respect to the fixed inertial reference *(F)*

$(\dot{\ })_F$ = the time derivative of a vector with respect to the fixed, inertial reference

$(\dot{\ })_{A,B,C,D}$ = the time derivative of a vector with respect to the noninertial $A,B,C,$ or D systems, respectively

Examples of Dimensionless Parameters

ρ'_i = $\rho_i/\mathcal{L}_e, i=1,2; I'_{Bi}=I_{Bi}/\mu\mathcal{L}_e^2, i=1,2,3$

c''_{B_i} = $c_{B_i}/\mu\mathcal{L}_e^2 s^2, i=1,2,3$

k''_{B_i} = $k_{B_i}/\mu\mathcal{L}_e^2 s, i=1,2,3$

k''_1 = $k_1/\mu s^2$-1

k''_2 = $k_2/\mu s$

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I. Introduction

ARTIFICIAL gravity in a space station system may be created in two different ways: the station in a rim-like configuration may be rotated about its axis of symmetry, or the station connected to a counterweight by a taut cable can be rotated about the system center of mass. The second system may have certain weight and power system advantages over the first; to change the spin-rate of the system it is necessary to adjust the effective equilibrium length of the cable, whereas in the rim configuration an active power source is required.

In an earlier treatment Paul[1] considered the planar motion and stability of a gravity-gradient stabilized, extensible dumbbell satellite system where the cable mass effects were neglected. He developed stability criteria and showed that, if the internal friction resulted from "material damping" within the elastic cable, there would be relatively little damping of a viscous nature, but that a nonlinear time-independent type of hysteretic damping could be significant. Bainum et al.[2] included the effects of distributed (unsymmetrical) end masses for the case of a gravitationally stabilized tethered-connected interferometer system and concluded that a combination of tether system damping and rotational damping of the motion of the end masses about their own mass centers must be employed to achieve adequate damping in each mode. The first three-dimensional stability analysis of a connected gravitationally stabilized system was presented by Robe.[3] His system consisted of two identical but unsymmetrical distributed end masses connected by a massless, extensible tether, resulting in nine degrees-of-freedom. It was shown that there is a decoupling of the small-amplitude motions within the orbital plane from those outside the plane; therefore, additional "out-of-plane" stability criteria, would, in general, have to be satisfied. Beletskii and Novikova[4] considered domains of possible three-dimensional motion for a gravitationally stabilized point-mass system connected by a flexible, massless tether for the cases of both a taut and a slack tether.

In the area of rotating connected systems Chobotov[5] included the effects of cable mass and elasticity with point mass end masses and two-dimensional motion. It was found that the gravity-gradient effects upon the small amplitude vibration stability of the rotating system are very small, and that the stability criteria are functions of the cable natural frequencies, the angular velocities of the station and orbital motion, and viscous damping parameters. Subsequently, Stabekis and Bainum[6] examined the motion and stability of a rotating space station-massless cable-counterweight configuration where the motion was restricted to the orbital plane. Although the system remained stable in the absence of rotational damping (of end body motions), this damping in addition to cable damping is necessary to achieve reasonable time constants for the nominal parameters considered. A paper by Nixon[7] deals with determining the dynamic equilibrium states in three dimensions for a completely undamped system with an arbitrary number of cables, but does not consider an analytical stability analysis. Anderson,[8] whose system had distributed end masses with lateral oscillations for three-dimensional motion, used an energy approach to analyze the motion of the system under the influence of disturbance torques. He found that the basic attitude response of the space station is that of an undamped second-order system and that coupled to this response are rigid body characteristics and cable lateral mode effects.

Of interest in this investigation is an examination of the three-dimensional motion of the rotating cable-connected system for the general case where the end bodies have a distributed mass (finite, unequal moments of inertia) and the possibility of energy dissipation in both the cable system and end bodies is included. To date, this treatment has not appeared in the open literature and would represent an extension to the problem considered in Ref. 6. A more comprehensive treatment of the work presented here is available in Ref. 9, the final report to NASA, the granting agency which supported this research.

II. Description of Mathematical Model

It is assumed that the system center of mass follows a circular orbit, that the cable is extensible but massless, and that the system equilibrium has a nominal spin rate in the orbit plane about an axis passing through its center of mass. Five different coordinate systems describe the motion. The fixed inertial reference (F) is located at the center of mass of the Earth, whereas, the A coordinate system is located at the center of mass of the system model with the A_1 axis along the local vertical, the A_2 axis in the direction of the velocity of the orbit, and the A_3 axis normal to the orbit plane. The B system is fixed in the space station (body 1 as shown in Fig. 1) at its center of mass. The axes of the B system are assumed to be the principal axes of body 1 with the cable attached at a point on the B_1 axis. A one-two-three sequence of rotations, respectively, is assumed to orient the B system with respect to the A system. The C system fixed in a body 2 (the counterweight) at its center of mass is defined the same way as the B system. Lastly, the D coordinate system is located at the center of mass of the model and is defined by two rotations with respect to the A system: an angle θ_1 in the orbit plane and then an angle θ_2 out of the plane. By these rotations the D_1 axis is parallel to the cable line.

The transformations from the A to the B systems, and from the A to the D systems are given in Eqs. (1) and (2) as:

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} c\beta_3 & s\beta_3 & 0 \\ -s\beta_3 & c\beta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta_2 & 0 & -s\beta_2 \\ 0 & 1 & 0 \\ s\beta_2 & 0 & c\beta_2 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta_1 & s\beta_1 \\ 0 & -s\beta_1 & c\beta_1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} c\theta_2 & 0 & s\theta_2 \\ 0 & 1 & 0 \\ -s\theta_2 & 0 & c\theta_2 \end{bmatrix} \begin{bmatrix} c\theta_1 & s\theta_1 & 0 \\ -s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{2}$$

where c and s indicate cosine and sine functions, respectively. The transformation between the A and C systems is similar to Eq. (1) with β_i replaced by γ_i.

III. Development of the Equations of Motion

A. Energy Expressions and the Rayleigh Dissipation Function

To use Lagrange's equations, it is necessary to express the total kinetic and potential energy of the system in terms of the variables which describe the system's motion. This can be performed by means of vector equations whose components are functions of the variables.

The equation for the translational kinetic energy is

$$T_T = \frac{1}{2} m_1 \dot{r}^2_{1/0} + \frac{1}{2} m_2 \dot{r}^2_{2/0} \tag{3}$$

where $\dot{r}^2_{i/0} = \dot{\bar{r}}_{i/0} \cdot \dot{\bar{r}}_{i/0}$ for $i = 1,2$. From the fact that the A system has a noninertial rotation, Eqs. (4) and (5) may be obtained:

$$(\dot{\bar{r}}_{1/0})_F = (\dot{\bar{r}}_{1/A})_F + \bar{\omega}_{A/F} \times \bar{r}_{A/0} \tag{4}$$

$$(\dot{\bar{r}}_{2/0})_F = (\dot{\bar{r}}_{2/A})_F + \bar{\omega}_{K/N} \times \bar{r}_{A/0} \tag{5}$$

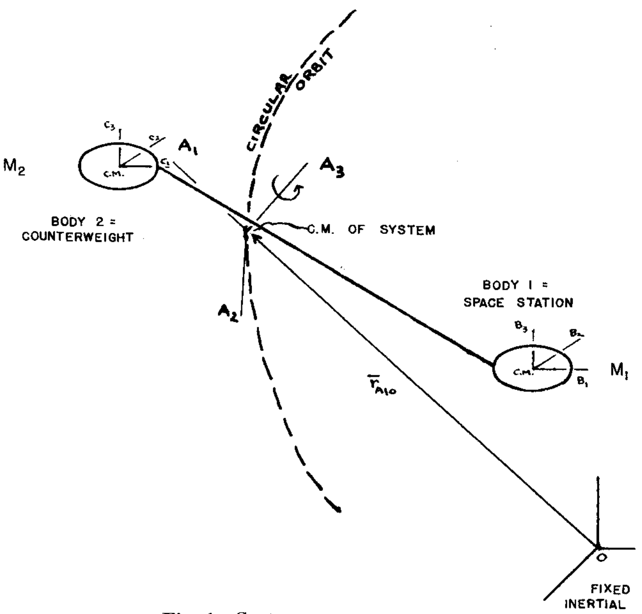

Fig. 1 System geometry.

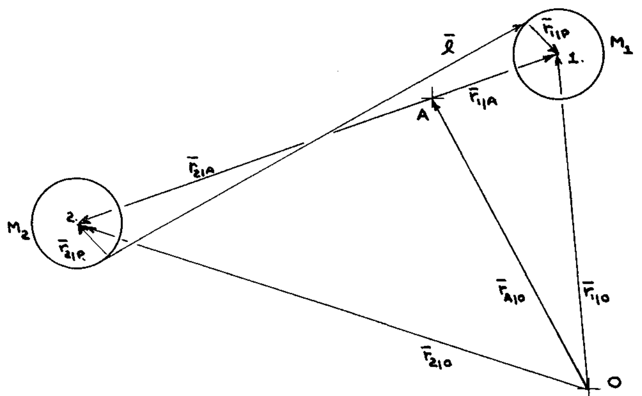

Fig. 2 Definition of vectors.

From the geometry of the system (Fig. 2), we deduce

$$\bar{\mathcal{L}} + \bar r_{1/P} - \bar r_{1/A} + \bar r_{2/A} - \bar r_{2/P} = 0 \tag{6}$$

From the fact that point A is the system cm,

$$m_1 \bar r_{1/A} + m_2 \bar r_{2/A} = 0 \tag{7}$$

The model used has two end masses. This indicates that for general unconstrained motion twelve variables are needed. If in addition, the variables θ_1, θ_2, and $\mathcal{L}$ are added for the cable line, there will be fifteen variables in all. Equations (6) and (7) can be used to express $\bar r_{1/A}$ and $\bar r_{2/A}$ in terms of the other vectors. Since the x,y,z coordinates of each end mass are then eliminated, Eqs. (6) and (7) can be considered as six equations of constraint. Thus nine independent variables describe the motion of the model and are selected as follows: θ_1 and θ_2 for the orientation of the cable with respect to the A system: $\mathcal{L}$ for the variable cable length; β_1, β_2, β_3 for the orientation of body 1 and γ_1, γ_2, γ_3, for the orientation of body 2.

Equations (6) and (7) can be combined to express the vectors $\bar r_{1/A}$ and $\bar r_{2/A}$ as

$$\bar r_{1/A} = \frac{m_2}{m_1+m_2}[\bar{\mathcal{L}} + \bar r_{1/P} - \bar r_{2/P}] \tag{8}$$

$$\bar r_{2/A} = -\frac{m_1}{m_1+m_2}[\bar{\mathcal{L}} + \bar r_{1/P} - \bar r_{2/P}] \tag{9}$$

The derivatives of Eqs. (8) and (9) are taken, noting that

$$\bar u = (\dot{\bar{\mathcal{L}}})_F + (\dot{\bar r}_{1/P})_F - (\dot{\bar r}_{2/P})_F \tag{10}$$

Then, after substituting the results of this differentiation into Eqs. (4) and (5), the following is obtained:

$$(\dot{\bar r}_{1/0})_F = \frac{m_2 \bar u}{m_1+m_2} + \bar\omega_{A/F}\times\bar r_{A/0} \tag{11}$$

$$(\dot{\bar r}_{2/0})_F = -\frac{m_1 \bar u}{m_1+m_2} + \bar\omega_{A/F}\times\bar r_{A/0} \tag{12}$$

It should be noted also that

$$\bar u = (\dot{\bar r}_{1/A})_F - (\dot{\bar r}_{2/A})_F = (\dot{\bar r}_{1/0})_F - (\dot{\bar r}_{2/0})_F \tag{13}$$

Equations (11) and (12) when substituted into Eq. (3) yield:

$$T_T = \frac{1}{2}\mu(\bar u\cdot\bar u) + \frac{1}{2}(m_1+m_2)|\bar\omega_{A/F}\times\bar r_{A/0}|^2 \tag{14}$$

where the second term is constant for a circular orbit. The rotational kinetic energy has the form

$$T_R = \frac{1}{2}\sum_{i=1}^{3}(I_{B_i}\omega^2_{B_i} + I_{C_i}\omega^2_{C_i}) \tag{15}$$

since the axes of the B and C systems are the principal axes of bodies 1 and 2, respectively.

The potential energy is, allowing for restoring forces in the cable and end bodies,

$$V = -\frac{GM_e(m_1+m_2)}{r_{A/0}} + k_1(\mathcal{L}-\mathcal{L}_0)^2/2 + \frac{1}{2}\sum_{i=1}^{3}(c_{B_i}\bar\beta_i^2 + c_{C_i}\bar\gamma_i^2) \tag{16}$$

where G is the gravitational constant, M_e is the mass of the Earth and the bars indicate variational variables defined later. The interaction of on-board magnetics with the ambient magnetic field could provide rotational restoring torques derivable from this type of potential.[2] It is further assumed in connection with Eq. (16) that the restoring force in the cable is proportional to the linear extension from the unstretched length—i.e., the cable obeys Hooke's Law.

Damping forces are assumed to act through a dissipation function of the form:

$$\mathcal{F} = \frac{1}{2}\left[\sum_{i=1}^{3}(k_{B_i}\dot{\bar\beta}_i^2 + k_{C_i}\dot{\bar\gamma}_i^2) + k_2\dot{\mathcal{L}}^2\right] \tag{17}$$

where $\dot{\bar\beta}$ and $\dot{\bar\gamma}_i$ are the time rates of change of the variational variables corresponding to β_i and γ_i. This type of damping on the end bodies could be accomplished by employing three orthogonal magnetometers together with appropriate electronics.[2]. It is also assumed that the cable will provide viscous damping proportional to the cable rate of extension.

B. Examples of Terms Expressed in the Nine Variables

The angular velocity of a body is expressible in terms of the angles used to describe the transformation matrix and the time rate of change of these angles. From the transformation matrix, Eq. (1),

$$\begin{aligned}\bar\omega_{B/A} = {} & (\dot\beta_1\cos\beta_3\cos\beta_2 + \dot\beta_2\sin\beta_3)\hat b_1 \\ & + (-\dot\beta_1\sin\beta_3\cos\beta_2 + \dot\beta_2\cos\beta_3)\hat b_2 \\ & + (\dot\beta_1\sin\beta_3 + \dot\beta_3)\hat b_3\end{aligned} \tag{18}$$

It is noted that $\dot{\beta}_1$ corresponds to a rotation about the A_1 axis, $\dot{\beta}_2$ corresponds to a rotation about the displaced A_2 axis, and $\dot{\beta}_3$ is associated with a rotation about the B_3 axis.

The expression for $\bar{u}$, the relative translational velocity vector, Eq. (10) may be expanded using the general equation relating the derivative of a vector in an inertial system to its derivative in a noninertial system,

$$(\dot{\bar{\mathcal{L}}})_F=(\dot{\bar{\mathcal{L}}})_D+\bar{\omega}_{D/F}\times\bar{\mathcal{L}} \tag{19}$$

$$(\dot{\bar{r}}_{1/P})_F=(\dot{\bar{r}}_{1/P})_B+\bar{\omega}_{B/F}\times\bar{r}_{1/P} \tag{20}$$

$$(\dot{\bar{r}}_{2/P})_F=(\dot{\bar{r}}_{2/P})_C+\bar{\omega}_{C/F}\times\bar{r}_{2/P} \tag{21}$$

Since the cable can stretch, $(\dot{\bar{\mathcal{L}}})_D=\dot{\mathcal{L}}\hat{d}_1$. Furthermore, $(\dot{\bar{r}}_{1/P})_B=(\dot{\bar{r}}_{2/P})_C=0$ since $\bar{r}_{1/P}$ and $\bar{r}_{2/P}$ are constant vectors in the B and C systems, respectively. For the case of planar motion, Eq. (10) can be reduced to the following:

$$\begin{aligned}\bar{u}=&[\dot{\mathcal{L}}-\rho_1(\dot{\beta}_3+\Omega)\sin(\beta_3-\theta_1)-\rho_2(\dot{\gamma}_3+\Omega)\\&\times\sin(\gamma_3-\theta_1)]\hat{d}_1+[(\dot{\theta}_1+\Omega)\mathcal{L}+\rho_1(\dot{\beta}_3+\Omega)\\&\times\cos(\beta_3-\theta_1)+\rho_2(\dot{\gamma}_3+\Omega)\cos(\gamma_3-\theta_1)]\hat{d}_2\end{aligned} \tag{22}$$

By comparing this with Eq. (8) of Ref. 6 we can relate the coordinates $\phi_{1,2}$ used there to the variables β_3 and γ_3 for the case of planar motion:

$$\beta_3=\theta_1+\phi_1;\quad \gamma_3=\theta_1+\phi_2 \tag{23}$$

for planar motion. The angles ϕ_1 and ϕ_2 describe the orientation between the cable line and the principal axis of each end body which is aligned with the attachment arm vector (Fig. 1 of Ref. 6).

C. Lagrange's General Equation and the Procedure for Developing the Equations of Motion

With all expressions, kinetic, potential energies, etc., in terms of the nine generalized coordinates, Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial(T-V)}{\partial q_j}=-\frac{\partial \mathfrak{F}}{\partial \dot{q}_j} \tag{24}$$

yield the nine equations of motion for this space station-counterweight system.

The nine equations of motion were derived with the approximation that $\sin(q_p)\approxeq q_p$ and $\cos(q_p)\approxeq 1$ where q_p is any one of $\beta_1,\beta_2,\theta_2,\gamma_1$, or γ_2 (angles out of the orbit plane). The approximation was made after the final differentiation and all terms of degree higher than two in q_p were considered small. In addition it was assumed that the attachment arm vectors $\bar{r}_{1/P}$, $\bar{r}_{2/P}$ are in the direction of the unit vectors, $\hat{b}_1$ and $\hat{c}_1$, respectively; for the more general case where the cable attachment point is not located on the $B_1(C_1)$ principal axis, the equations would have to be appropriately modified.

In a stability analysis concerned with motion about an equilibrium state, variables are used which measure the deviation from the equilibrium motion. The following definitions were used in this respect:

$$\theta_1=\dot{\theta}_n t+\chi\qquad \delta=\frac{\mathcal{L}-\mathcal{L}_e}{\mathcal{L}_e}$$

$$\beta_3=\dot{\theta}_n t+\alpha_1\quad \gamma_3=\dot{\theta}_n t+\alpha_2 \tag{25}$$

where $\dot{\theta}_n t$ is the equilibrium value of θ_1,β_3, and γ_3 for any time, t, and $\mathcal{L}_e$ is the equilibrium cable length to be determined. χ, α_1,α_2, and δ are the new variational coordinates corresponding to θ_1,β_3, γ_3, and $\mathcal{L}$, the original variables. The original out-of-plane angles are zero at equilibrium and accordingly serve as variational coordinates. Thus, referring to Eqs. (16) and (17), $\tilde{\beta}_3=\alpha_1,\tilde{\gamma}_3=\alpha_2$, and $\tilde{\beta}_i=\beta_i(i=1,2)$, $\tilde{\gamma}_i=\gamma_i(i=1,2)$. The variational form of the θ_1 equation in which terms of the degree higher than two in the variational variables have been neglected is shown in Ref. 9.

D. Linearization of the Equations of Motion

After examining the variational form of the equations, the linear equations were obtained by making use of trigonometric identities and neglecting all second and higher degree terms in the variational variables. The approximations:

$$\sin(\beta_3-\theta_1)=\sin(\alpha_1-\chi)\approx\alpha_1-\chi$$

and

$$\cos(\beta_3-\theta_1)=\cos(\alpha_1-\chi)\approx 1,$$

etc., also yield linear terms. The form of the coefficients of the variables in the linear equations is simplified by nondimensionalizing time as follows:

$$\tau=(\dot{\theta}_n+\Omega)t \tag{26}$$

Differentiation with respect to τ is indicated by super primes.

The resulting linearized equations, in nondimensional form are shown (noting that primes above the various coefficients represent nondimensionalization of physical parameters and should not be confused with differentiation):

$$\begin{aligned}\chi\text{--}\ &\chi''+\rho'_1\alpha''_1+\rho'_2\alpha''_2+2\delta'+(\rho'_1+\rho'_2)\chi\\&-\rho'_1\alpha_1-\rho'_2\alpha_2=0\end{aligned} \tag{27a}$$

$$\delta\text{--}\ \delta''-2\chi'-2\rho'_1\alpha'_1-2\rho'_2\alpha'_2+k''_1\delta+k''_2\delta'=0 \tag{27b}$$

$$\begin{aligned}\alpha_1\text{--}\ &(\rho'^2_1+I'_{B_3})\alpha''_1+\rho'_1\rho'_2\alpha''_2+\rho'_1\chi''+k''_{B_3}\alpha'_1\\&+2\rho'_1\delta'-\rho'_1\chi+\rho'_1(1+\rho'_2)\alpha_1\\&+c''_{B_3}\alpha_1-\rho'_1\rho'_2\alpha_2=0\end{aligned} \tag{27c}$$

$$\begin{aligned}\alpha_2\text{--}\ &(\rho'^2_2+I'_{C_3})\alpha''_2+\rho'_1\rho'_2\alpha''_1+\rho'_2\chi''+k''_{C_3}\alpha'_2\\&+2\rho'_2\delta'-\rho'_2\chi+\rho'_2(1+\rho'_1)\alpha_2\\&+c''_{C_3}\alpha_2-\rho'_1\rho'_2\alpha_1=0\end{aligned} \tag{27d}$$

$$\begin{aligned}\beta_2\text{--}\ &\frac{1}{2}(I'_{B_2}+I'_{B_1})\beta''_2+[(I'_{B_2}+I'_{B_1})\Gamma-I'_{B_3}]\beta'_1\\&+k''_{B_2}\beta'_2+[c''_{B_2}+I'_{B_3}\Gamma-\frac{1}{2}(I'_{B_1}\\&+I'_{B_2})\Gamma^2]\beta_2=0\end{aligned} \tag{28a}$$

$$\begin{aligned}\beta_1\text{--}\ &\frac{1}{2}(I'_{B_2}+I'_{B_1})\beta''_1+[I'_{B_3}-(I'_{B_2}-+I'_{B_1})\Gamma]\beta'_2\\&+k''_{B_1}\beta'_1+[c''_{B_1}+I'_{B_3}\Gamma-\frac{1}{2}(I'_{B_1}\\&+I'_{B_2})\Gamma^2]\beta_1=0\end{aligned} \tag{28b}$$

$$\theta_2\text{--}\ \theta''_2+(1+\rho'_1+\rho'_2)\theta_2=0 \tag{28c}$$

$$\begin{aligned}\gamma_2\text{--}\ &\frac{1}{2}(I'_{C_2}+I'_{C_1})\gamma''_2+[(I'_{C_2}+I'_{C_1})\Gamma-I'_{C_3}]\gamma'_1\\&+k''_{C_2}\gamma'_2+[c''_{C_2}+I'_{C_3}\Gamma-\frac{1}{2}(I'_{C_1}+I'_{C_2})\Gamma^2]\gamma_2=0\end{aligned} \tag{28d}$$

$$\begin{aligned}\gamma_1\text{--}\ &\frac{1}{2}(I'_{C_2}+I'_{C_1})\gamma''_1+[I'_{C_3}-(I'_{C_2}+I'_{C_1})\Gamma]\gamma'_2\\&+k''_{C_1}\gamma'_1+[c''_{C_1}+I'_{C_3}\Gamma-\frac{1}{2}(I'_{C_1}+I'_{C_2})\Gamma^2]\gamma_1=0\end{aligned} \tag{28e}$$

The linear equations separate into four parts. The in-plane Eqs. (27a and d) are completely uncoupled from the out-of-plane equations. The out-of-plane equations are made up of three separate parts: 1) the β_2, β_1 Eqs. (28a and b) which are dynamically coupled, 2) the θ_2 Eq. (28c), and 3) the γ_2,γ_1 equations, (28d and e) also dynamically coupled. The β_2,β_1 equations apply to body 1 in the same way that the γ_2,γ_1 equations apply to body 2. The θ_2 equation, however, separates completely from the other equations and indicates simple harmonic motion. Although asymptotic stability of the system clearly will not occur for the case in which the cable has an out-of-plane perturbation, mission requirements could perhaps still be accomplished for small disturbances. From the definition of the angle θ_2, the system would achieve an equilibrium motion in a plane slightly inclined with respect to the original plane of motion, but with the same equilibrium length and nominal spin-rate. For such a situation, all stability criteria previously developed by considering the original equilibrium motion would apply to the new equilibrium motion.[10]

At equilibrium, the values of the variational coordinates are zero. An examination of the $\mathcal{L}$ equation at equilibrium yields the equilibrium condition:

$$(k''_1+1)\delta_0=-(1+\rho'_1+\rho'_2) \tag{29}$$

which states that the centrifugal force due to the inertial rotation of the system is balanced by the tension in the cable. Eq. (29), after conversion to dimensional form, can be written:

$$\mathcal{L}_e=\frac{k_1\mathcal{L}_0+\mu(\rho_1+\rho_2)(\dot\theta_n+\Omega)^2}{k_1-\mu(\dot\theta_n+\Omega)^2} \tag{30}$$

The condition $k_1\leqslant\mu(\dot\theta_n+\Omega)^2$ implies that $\mathcal{L}_e$ is either negative or infinite. For realistic values of $\mathcal{L}_e$, it is necessary that k_1 be greater than $\mu(\dot\theta_n+\Omega)^2$, since if this condition is not satisfied the cable will not provide sufficient tension to balance the centrifugal force of the spin.

The in-plane equations can be compared with those in Ref. 6 where the analysis was confined to the orbit plane. From Eqs. (23) and the definition of the variational variables, Eqs. (25), one can relate the $\phi_{1,2}$ variational coordinates of Ref. 6 with those used in the present analysis:

$$\alpha_1=\chi+\phi_1 \quad \alpha_2=\chi+\phi_2 \tag{31}$$

The in-plane linear equations can thus be written in terms of the variables defined by Stabekis and Bainum.[6]

After using Eq. (31) and manipulating the in-plane equations, Eqs. (27a-d), it was found that Eq. (16) of Ref. 6 did not include the linear terms $\mu\rho_1\ddot\phi_1(\mathcal{L}+\rho_1+\rho_2)$, and $\mu\rho_2\ddot\phi_2(\mathcal{L}+\rho_1+\rho_2)$ while in Eq. (17), the terms $-2\rho_1(\dot\theta_n+\Omega)\dot\phi_1$, and $-2\rho_2(\dot\theta_n+\Omega)\dot\phi_2$ were missing. The equations of motion for the system used by Ref. 6 were completely rederived using the variables defined therein and confirmed this result. When converted to nondimensional form, these four terms all contain the coefficient $\rho_i/\mathcal{L}_e,<<<1$ for the examples considered (attachment arm lengths much shorter than cable equilibrium length). Thus the effect of neglecting these terms on the numerical results previously reported[6] would be expected to be small.

IV. Stability Analysis

A. General Stability Considerations

The procedure used in analyzing the stability of the general system was to first obtain the characteristic polynominal. The coefficients of this polynominal can be considered in conjunction with the Routh-Hurwitz necessary-sufficient conditions for the roots to have negative real parts, to provide conditions for stability. The in-plane and out-of-plane criteria can be obtained independently of each other because of the decoupling of the equations.

Considering first the in-plane equations, Eqs. (27a-d), a characteristic equation can be developed in the following form:

$$\sum_{I+0}^{8} a_i\lambda^{8-i}=0 \tag{32}$$

The coefficients appearing in Eq. (32) are functions of the system parameters and are given explicitly in Ref. 9. Because of the complexity of the coefficients in Eq. (32), the necessary and sufficient Routh-Hurwitz criteria were not developed for the general case; nevertheless, analytic stability criteria were developed for special cases, and for the general case, by considering the signs of selected coefficients in the characteristic equation.

The necessary condition for stability is that all of the coefficients in the characteristic equation have the same sign and be nonzero. By inspection of the coefficients in Eq. (32), it is seen that in-plane stability is insured if at least one of the following forms of damping is present: cable damping (k''_2), or rotary damping in at least one end body about an axis nominally perpendicular to the plane of rotation (k''_{B_3} or k''_{C_3}). Under this condition all of the odd coefficients wil be positive nonzero. From consideration of a_8, which can be related to the system restoring constants as follows:

$$a_8=k''_1[\rho'_1+\rho'_2)c''_{B_3}c''_{C_3}+\rho'_1\rho'_2(1+\rho'_1+\rho'_2)(c'_{B_3}+c'_{C_3})] \tag{33}$$

a restoring torque capability must be present on at least one of the end bodies about an axis perpendicular to the nominal plane of rotation—either c_{B_3} or $c_{C_3}>0$. Also for $a_8>0$, k''_1 must be greater than zero. Thus it must be true that k_1 be greater than $\mu(\dot\theta_n+\Omega)^2$ to allow for the possibility of stability—in agreement with the results obtained from the equilibrium condition [viz., discussion in connection with Eqs. (29) and (30)].

The out-of-plane equations (only the equations for one end body need be considered, e.g. Eqs. (28a and b), since the equations for the second end body are analogous to the equations for the first end body) yield the following characteristic equation:

$$m_{11}\lambda^4+m_{11}(k''_{B_1}+k''_{B_2})\lambda^3+[m_{11}(K_{11}+K_{22})+k''_{B_1}k''_{B_2}+C_{12}^2]\lambda^2+(k''_{B_1}K_{22}+k''_{B_2}K_{11})\lambda+K_{11}K_{22}=0 \tag{34}$$

where

$$m_{11}=\frac{1}{2}(I'_{B_2}+I'_{B_1});\ C_{12}=I'_{B_3}-(I'_{B_2}+I'_{B_1})\Gamma$$

$$K_{11}=c''_{B_1}+I'_{B_3}\Gamma-\frac{1}{2}(I'_{B_2}+I'_{B_1})\Gamma^2$$

$$K_{22}=c''_{B_2}+I'_{B_3}\Gamma-\frac{1}{2}(I'_{B_2}+I'_{B_1})\Gamma^2$$

The four necessary and sufficient Routh-Hurwitz stability criteria associated with Eq. (34) can be developed to yield:

$$k''_{B_1}+k''_{B_2}>0 \tag{35}$$

$$m_{11}k''_{B_1}c''_{B_1}+m_{11}k''_{B_2}c''_{B_2}+(k''_{B_1}+k''_{B_2})(k''_{B_1}k''_{B_2}+(I'_{B_3}-\frac{1}{2}(I'_{B_2}+I'_{B_1})\Gamma)^2+m^2_{11}\Gamma^2)$$

$$>m_{11}(k''_{B_1}+k''_{B_2})(I'_{B_3}-\frac{1}{2}(I'_{B_2}+I'_{B_1})\Gamma)\Gamma \qquad (36)$$

$$m_{11}k''_{B_1}k''_{B_2}(c''_{B_1}-c''_{B_2})^2+[k''^2_{B_2}c''_{B_1}+k''^2_{B_1}c''_{B_2}$$
$$+(k''_{B_1}+k''_{B_2})^2(I'_{B_3}-\frac{1}{2}(I'_{B_2}+I'_{B_1})\Gamma)\Gamma$$
$$+k''_{B_1}k''_{B_2}(c''_{B_1}+c''_{B_2})]\ [k''_{B_1}k''_{B_2}+(I'_{B_3}$$
$$-(I'_{B_2}+I'_{B_1})\Gamma)^2]>0 \qquad (37)$$

$$[c''_{B_1}+(I'_{B_3}-\frac{1}{2}(I'_{B_2}+I'_{B_1})\Gamma)\Gamma][c''_{B_2}$$
$$+(I'_{B_3}-\frac{1}{2}(I'_{B_2}+I'_{B_1})\Gamma)\Gamma]>0 \qquad (38)$$

From Condition (35), there must be rotational damping on each of the end bodies about at least one of the principal axes which, at equilibrium, will lie in the plane of rotation. (Note that rotational damping must be present on both end bodies since similar criteria may be developed for the second end body.) Furthermore, from Condition (36), if

$$c''_{B_i}\geq I'_{B_3}\Gamma-\frac{1}{2}(I'_{B_2}+I'_{B_1})\Gamma^2 \text{ for both } i=1,2 \qquad (39)$$

asymptotic stability of the out of plane motion is assured if Condition (35) is also satisfied. It is also seen that the satisfaction of Condition (39) guarantees Condition (38).

B. Special Cases of the Linear In-Plane Equations

Assumptions on the physical model can reduce the complexity of the in-plane stability analysis. The cases of: identical end bodies; where the cable is attached to the cm of both end bodies; and point mass end masses are treated.

The case of completely identical end masses

For this case, it was assumed that the space station-counterweight system was completely identical, that is, that $\rho_1=\rho_2$, $m_1=m_2$, and for $i=1,2,3$, $I_{B_i}=I_{C_i}$, $k_{B_i}=k_{C_i}$ and $c_{B_i}=c_{C_i}$. The characteristic equation separates into two factors for this case as shown below:

$$f(\lambda)\cdot g(\lambda)=0$$

where

$$f(\lambda)=I'_{B_3}\lambda^2+k''_{B_3}\lambda+\Delta+\rho'^2_1$$
$$R=c''_{B_3}+\rho'_1(1+\rho'_1) \qquad (40)$$

and $g(\lambda)$ is a sixth-degree polynominal in λ given explicitly in Ref. 9. The quadratic factor, $f(\lambda)$, indicates roots given by

$$\lambda=-\frac{k''_{B_3}}{2I'_{B_3}}\pm\frac{1}{2I'_{B_3}}[k''^2_{B_3}-4I'_{B_3}(R+\rho'^2_1)]^{1/2} \qquad (41)$$

In-plane instability is associated with this mode for $k''_{B_3}\leq 0$ or $c''_{B_3}\leq-\rho'_1(1+2\rho'_1)$. However, it should be recalled from consideration of Eq. (32) that either c_{B_3} or $c_{C_3}>0$ is required for stability of the general system—i.e., a stronger criterion on the restoring constant than is apparent from Eq. (41). The fact that the missing terms of Ref. 6 are not involved in this factor makes the results obtained analogous to those obtained therein.

The case of zero attachment arms

When $\rho_1=\rho_2=\rho'_1=\rho'_2=0$ the cable is attached at the cm's of both end bodies and the in-plane characteristic equation has the form:

$$\lambda^2(\lambda^2+k''_2\lambda+k''_1+4)$$
$$\times(I'_{B_3}\lambda^2+k''_{B_3}\lambda+c''_{B_3})$$
$$\times(I'_{C_3}\lambda^2+k''_{C_3}\lambda+c''_{C_3})=0 \qquad (42)$$

The repeated zero root resulting from Eq. (42) is indicative that in-plane asymptotic stability never occurs for this case.[9] But aside from this, stability is indicated in the other modes for $c''_{B_3}>0$ and $k''_{B_3}>0$; $c''_{C_3}>0$ and $k''_{C_3}>0$. These results indicate that rotational damping and restoring effects must be present on both end bodies, as well as cable damping and restoring forces. This is a stronger criterion for in-plane stability than that deduced earlier in connection with the sign of the coefficients in Eq. (32). For an actual situation where the attachment arms are much shorter than the cable length $\rho'_i=\rho'_i/\mathcal{L}_e\approx 0$, the results obtained here would have important implications on stability.

The case of point end masses

For this case, $\rho'_1=\rho'_2=0$ and there is no rotational end body motion so that only the $\mathcal{L}$ and θ_1 equations remain. The in-plane characteristic equation will contain only the terms: $\lambda^2(\lambda^2+k''_2\lambda+k''_1+4)$ and the repeated zero roots still will occur indicating the presence of a first integral of the motion.

C. Special Cases of the Out-of-Plane Linear Equations

The out-of-plane equations, yielding a fourth degree characteristic equation, allowed the determination of the Routh-Hurwitz conditions for the general system. The next most simple case is that for which the β_2 and β_1 equations uncouple ($C_{12}=0$). In this instance,

$$\frac{I_{B_3}}{(I_{B_2}+I_{B_1})}=\Gamma \qquad (43)$$

The β_2 and β_1 equations become two second-order equations from which it is clear that asymptotic stability occurs in these modes for

$$c''_{B_2} \text{ and } c''_{B_1}>-\frac{1}{2}I'_{B_3}\Gamma=-\frac{1}{2}(I'_{B_1}+I'_{B_2})\Gamma^2 \qquad (44)$$

with k''_{B_2} and k''_{B_1} both positive.

For a realistic system where the spin rate is much greater than the orbital rate, $\Gamma\ll 1$. Therefore Eq. (43) would be satisfied only by bodies having $I_{B_3}\ll I_{B_2}+I_{B_1}$ when the end bodies have their "long axis" perpendicular to the nominal plane of rotation (i.e., for long, slender pencil-shaped bodies).

V. Numerical Analysis

When the least damped mode of a system is examined, the worst possible response of the system is considered. If it can be determined for which set of system parameters the least damped mode decays faster, then the system damping can be optimized. The roots of the system characteristic equation can be calculated numerically for a specific set of system parameters. By incrementing the system parameters one at a time, a complete range of physical constants can be considered. Then by finding the least damped mode and plotting the time constant of this mode as a function of each parameter, the optimum set of system constants can be determined.

In this section, the numerical results obtained by the procedure outlined are compared with those results given in Ref. 6 for the in-plane characteristic equation. The out-of-plane modes are examined for the same range of parameters. A typical transient response of the linear system is also presented.

All computer results were obtained by means of an IBM 1130 computer system. The roots of the characteristic equations were calculated using a Newton-Raphson iteration technique. The transient responses were obtained by integrating the linear equations employing a Runge-Kutta fourth-order method. The inplane optimization program required about fifty minutes for each curve of T vs system parameter which contained 250 points. This can be compared with the out-of-plane optimization program which only used ten minutes of computer time for the same number of points. For the computational time step chosen ($\Delta t = 0.25$ simulated problem seconds), the transient responses required 30 sec for each simulated problem second.

In all computations, for both examples considered it was assumed

$$c''_{B_1} = c''_{B_2} = c''_{B_3} = c''_{C_1} = c''_{C_2} = c''_{C_3}$$

and

$$k''_{B_1} = k''_{B_2} = k''_{B_3} = k''_{C_1} = k''_{C_2} = k''_{C_3}$$

A. Identical End Bodies

The time constant of the least damped mode T was calculated as a function of c''_{B_i} and k''_{B_i} in this example. The following parameters remained constant at the values given below:

$\Omega =$ 0.055 deg/sec	$k_1 =$ 1000 lb/ft
$\dot{\theta}_n =$ 32.0 deg/sec	$k_2 =$ 56.7 lb-sec/ft
$m_1 = m_2 = 600$ slugs	$I_{B_1} = I_{C_1} = 81{,}000$ sl-ft^2
$\rho_1 = \rho_2 = 12$ ft	$I_{B_2} = I_{C_2} = 80{,}000$ sl-ft^2
$\mathcal{L}_0 =$ 230 ft	$I_{B_3} = I_{C_3} = 86{,}400$ sl-ft^2

In the first study, the rotational damping constant, k''_{B_3}, had the constant value of 15,500 ft-lb-sec/rad as the spring constant, c''_{B_3}, was incremented. Figure 3 shows that T increases slightly as c''_{B_3} increases. Also shown is the curve given in Ref. 6 in which a rotational restoring constant of 5000 ft-lb/rad corresponded to the minimum value of T for this same set of system constants. The corresponding out-of-plane graph appears in Fig. 4. Here, as c''_{B_1} increases, T decreases in a manner which seems asympotic. It can be seen that the minimum time constant achieved by Stabekis and Bainum[6] for the in-plane modes is about two orders of magnitude less than that obtained here for the same range of parameters. (It should be noted that in Fig. 3 and all subsequent figures, the parameter shown on the abscissa is dimensionless. For the nominal system parameters considered the conversion to the corresponding dimensional quantity is given below the abscissa on each figure.)

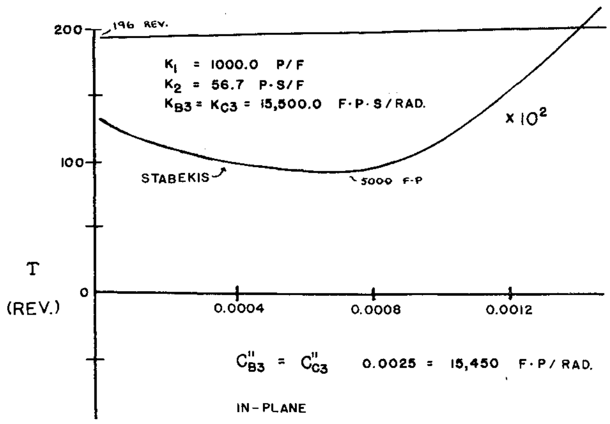

Fig. 3 T vs c''_{B_3}

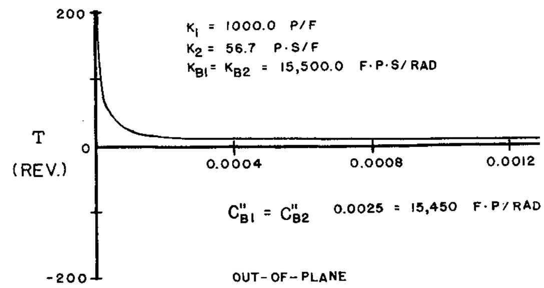

Fig. 4 T vs c''_{B_1}.

Throughout this section it is seen that the magnitudes of the optimum T from Ref. 6 were two and sometimes three orders of magnitude smaller than the present work. This difference was because rotational damping and restoring system constants of Ref. 6 were defined with respect to angles and angular rates measured from the cable line where, in this analysis, system constants were defined with respect to variational angles and angular rates which include the effect of χ and $\dot{\chi}$ [Eqs. (31)].

It should be recalled from the stability analysis of the general in-plane Eq. (32), that c_{B_3} (or c_{C_3}) > 0 is necessary for in-plane stability. For this reason, time constants associated with zero values of rotational restoring constants are not indicated for this unstable situation in Fig. 3.

Figure 5 shows T vs k''_{B_3} where c''_{B_3} was held constant at 5000 ft-lb/rad. Near $k''_{B_3} = 0$, the value of T is very large and as k''_{B_3} increases T decreases in asymptotic fashion. The out-of-plane case of Fig. 6 has the same characteristics as the in-

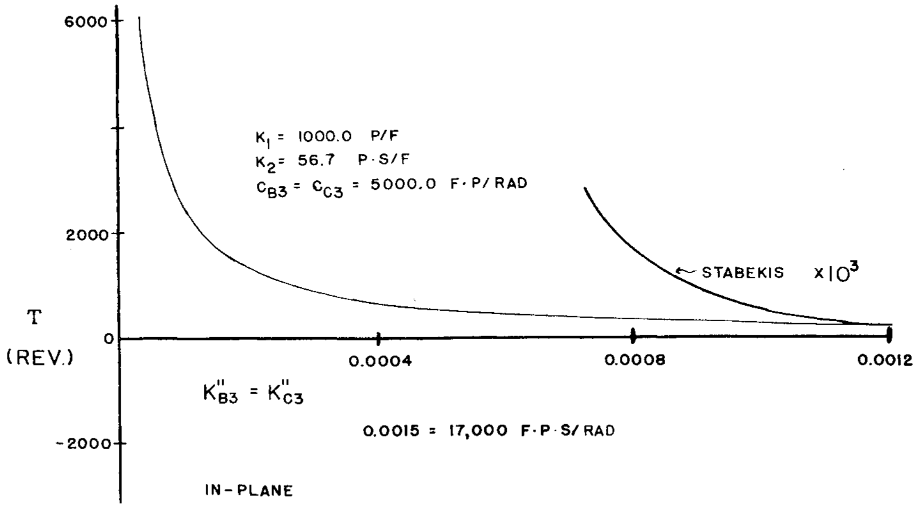

Fig. 5 T vs k''_{B_1}.

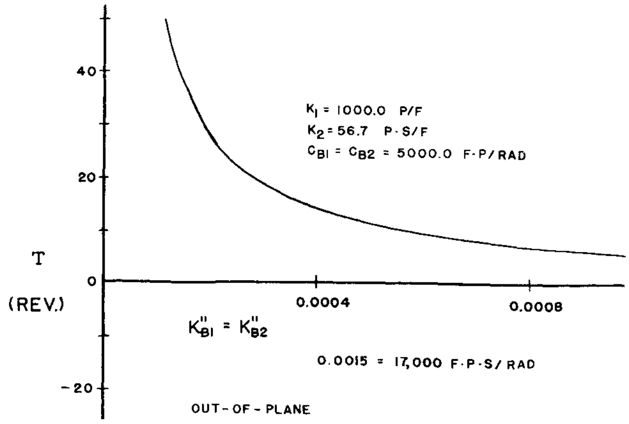

Fig. 6 T vs k''_{B_1}.

plane case but with a minimum time constant of about one order of magnitude smaller. Included in Fig. 5 are the results from Ref. 6 which show that T minimum is reached for a rotational damping constant of approximately 12,000 ft-lb-sec/rad.

A transient response of the identical system studied in Figs. 3-6 with $c_{B_i}=5000$ ft-lb/rad and $k_{B_i}=15{,}500$ ft-lb-sec/rad is shown given in Fig. 7. The initial conditions used were, zero velocities in all variables and,

$$\chi=0; \quad \mathcal{L}-\mathcal{L}_e=0.48 \text{ ft}; \quad \alpha_1=0.1 \text{ rad}$$

$$\alpha_2=-0.1 \text{ rad}; \quad \beta_1=0.1 \text{ rad}; \quad \beta_2=0$$

From Fig. 7, the initial high-frequency motion of the cable is seen to be greatly damped for the value of cable damping chosen ($k_2=56.7$ lb-sec/ft). For the parameters chosen, $\rho_1=\rho_2=12$ ft and $\mathcal{L}_e=256$ ft, ρ'_1 and ρ'_2 have the value of $0.0468 \ll 1$. The results of the special case of zero attachment arms indicate that the in-plane equations are weakly coupled for very small ρ'_1 and ρ'_2. Thus the response of χ shows predominately a lower frequency motion, and the initial response of $\mathcal{L}-\mathcal{L}_e$ shows mainly the motion associated with the cable, that is, until this motion decays and the effects of coupling become of the same order of magnitude. This same type of motion is apparent in the responses of α_1 and α_2. (The α_2 response is similar to that of α_1 except for a phase shift in the initial high frequency transient and is not included in Fig. 7.) The β_1 and β_2 curves indicate damping, but because the out-of-plane motion is independent of attachment arm lengths, the effects seen in the in-plane graphs do not take place. Since the motion of θ_2 is simple harmonic [see Eq. (28c)], its response is not shown.

The transient response was considered for 600 sec to reveal the damping of the lowest frequency motion. This simulation required about five hours of computer time. Assuming that the lower frequency motion of the in-plane responses is more representative of the least damped mode of motion, the time constant of this motion can be measured directly from the responses. The time constant of the χ response is measured to be ≈ 2186.37 sec, approximately 194.68 revolutions for the given inertial spin rate of this system (32.055 deg/sec). From Fig. 3, at $c_{B_3}=5000$ ft-lb/rad ($c''_{B_3}\approx 0.0008$) T the time constant of the least damped mode, is about 197 revolutions. The time constant obtained from the transient response is therefore about 1% different from the value predicted for the least damped mode. However, the accuracy in measuring time constants from the transient motion largely depends on the error obtained in measuring the amplitude ratios. From consideration of the other in-plane responses, it can be seen that the measured time constants are within ± 4 revolutions of that determined above.

The same procedure applied to the β_1 and β_2 motions reveals that the time constant of the β_1 response is about 4.35

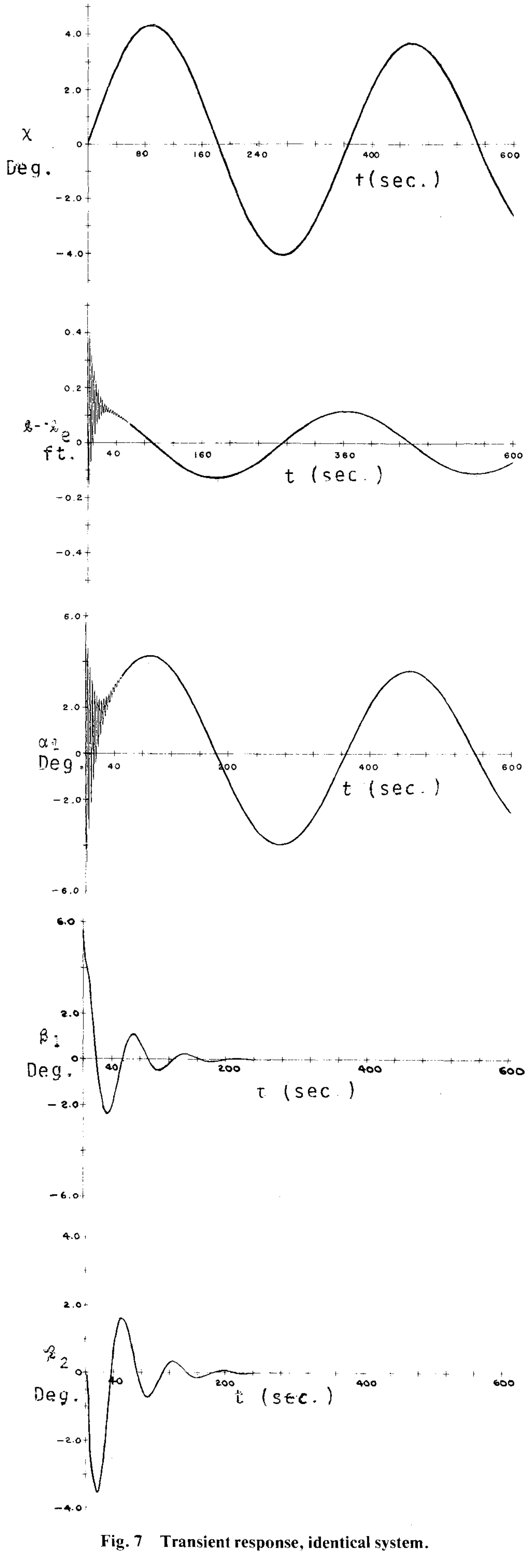

Fig. 7 Transient response, identical system.

revolutions and the time constant of the β_2 response is about 3.79 revolutions. Since these two motions are more highly coupled than the in-plane motions, the transient responses do not indicate the least damped mode to the same degree as the in-plane transient responses. Nevertheless, the time constants calculated are less than the time constant of the least damped mode ($T \approx 7$ rev) obtained from Fig. 4 at $c''_{B_1} \approx 0.0008$.

Based on the measured time constants of the transient responses of both in-plane and out-of-plane coordinates, it can be observed that these results are consistent with those predicted by an examination of the roots of the system characteristic equation. Additional results similar to those illustrated in Figs. 3-7 are presented for a case of unidentical end masses in Ref. 9. The effect of gravity-gradient torques on the small amplitude attitude motion is also studied and found to be small except in a resonant situation.[9,11]

VI. Conclusions

For the three-dimensional analysis of a rotating cable-connected space station system, the in-plane linear equations separate from the out-of-plane linear equations for small amplitude motion. For small perturbations on the cable's orientation out of the original plane of rotation, the system will tend to rotate in a plane inclined to the original plane of rotation without affecting the spin rate of the system. From the out-of-plane general stability criteria, positive damping is necessary about at least one principal axis on both end bodies in the plane of nominal rotation.

The cable restoring constant must be greater than the value of the reduced mass of the system multiplied by the square of the system's inertial spin rate for (in-plane) stability. From the necessary condition for in-plane stability, rotational restoring capability about an axis perpendicular to the nominal spin plane and on at least one end body is necessary for stability in the coordinates selected. For the case of identical end masses, positive damping and restoring torques about this same axis are necessary for stability.

For the special case in which the cable is attached at the center of mass of the end bodies, damping and restoring effects must be provided on both end masses about an axis normal to the plane of rotation. Damping and restoring capability proportional to angles and angular rates measured from the cable line provide time constants significantly lower than those resulting from damping and restoring with respect to the variational rates and angles.

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